

# NON-UNIQUENESS OF ERGODIC MEASURES WITH FULL HAUSDORFF DIMENSION ON A GATZOURAS-LALLEY CARPET

JULIEN BARRAL AND DE-JUN FENG

**ABSTRACT.** In this note, we show that on certain Gatzouras-Lalley carpet, there exist more than one ergodic measures with full Hausdorff dimension. This gives a negative answer to a conjecture of Gatzouras and Peres in [8].

## 1. INTRODUCTION

The problem we are interested in is the uniqueness of ergodic invariant measures on non-conformal repellers with full Hausdorff dimension (see [7, 3] for a survey). For  $C^{1+\alpha}$  conformal repellers, the existence and the uniqueness of an ergodic measure with full dimension follows from Bowen's equation together with the classical thermodynamic formalism [17].

For non-conformal repellers much less is known. The problem of existence of an ergodic measure with full dimension is solved for the class of Lalley-Gatzouras carpets and its nonlinear version [6, 11, 12]. In [8], Gatzouras and Peres conjectured that such a measure is unique. However, in this note, we show that this may fail on linear Lalley-Gatzouras carpets. Such a phenomenon is known for some other examples of self-affine sets constructed by Käenmäki and Vilppolainen [5].

To construct our example, let  $(X, \sigma_X)$  and  $(Y, \sigma_Y)$  be one-sided full shifts over finite alphabets  $\mathcal{A}$  and  $\mathcal{B}$ , respectively. Let  $\pi : X \rightarrow Y$  be a 1-block factor map, i.e., there is a map  $\tilde{\pi} : \mathcal{A} \rightarrow \mathcal{B}$  such that

$$\pi(x) = (\tilde{\pi}(x_i))_{i=1}^{\infty}, \quad x = (x_i)_{i=1}^{\infty} \in X.$$

Let  $\phi : X \rightarrow \mathbb{R}$  and  $\psi : Y \rightarrow \mathbb{R}$  be two positive functions which are constants over the cylinders of first generation of  $X$  and  $Y$  respectively, i.e.,

$$\phi(x) = \phi(x_1), \quad \psi(y) = \psi(y_1)$$

for each  $x = (x_i)_{i=1}^{\infty} \in X$  and  $y = (y_i)_{i=1}^{\infty} \in Y$ . Furthermore, assume that  $\phi(x) > \psi(\pi(x))$  for all  $x \in X$ .

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Define

$$(1.1) \quad P(\phi, \psi) = \sup \left\{ \frac{h_\mu(\sigma_X) - h_{\mu \circ \pi^{-1}}(\sigma_Y)}{\int \phi d\mu} + \frac{h_{\mu \circ \pi^{-1}}(\sigma_Y)}{\int \psi \circ \pi d\mu} \right\},$$

where the supremum is taken over the collection  $M(X, \sigma_X)$  of all  $\sigma_X$ -invariant Borel probability measures on  $X$ . Here  $h_\mu(\sigma_X)$  stands for the measure-theoretic entropy of  $\sigma_X$  with respect to  $\mu$  (cf. [17, 19]). Since the entropy maps  $\mu \mapsto h_\mu(\sigma_X)$  and  $\mu \mapsto h_{\mu \circ \pi^{-1}}(\sigma_Y)$  are upper semi-continuous, the supremum is attained on  $M(X, \sigma_X)$ . Moreover, since  $\phi(x)$  and  $\psi(y)$  only depend on the first coordinates of  $x$  and  $y$ , the supremum can be only attained at Bernoulli measures in  $M(X, \sigma_X)$ <sup>1</sup>.

In the next section, we construct an example to show that in the above general setting, there may have two different Bernoulli measures in  $M(X, \sigma_X)$  attaining the supremum in (1.1), which leads to a counter-example to Gatzouras and Peres conjecture on Lalley-Gatzouras carpets (see Section 3).

In fact, Gatzouras and Peres raised the wider conjecture claiming that if  $f$  is a smooth expanding map, then any compact invariant set  $K$  which satisfies specification carries a unique ergodic invariant measure  $\mu$  of full dimension. Moreover,  $\mu$  is mixing for  $f$ . This conjecture was proved to be true in some special cases, e.g., as we said when  $f$  is a conformal  $C^{1+\alpha}$  map on smooth Riemannian manifolds [8], and also when  $f$  is a linear diagonal endomorphism on the  $d$ -torus [4]. In particular, it is true for Bedford-McMullen self-affine carpets and sponges [2, 14, 9] and some sofic self-affine sets [18, 20, 15].

The same kind of questions have been studied on horseshoes. It is proved in [13] that for nonlinear horseshoes there may be no ergodic measure of full dimension, while such a measure exists for linear horseshoes [1], but may be not unique [16].

## 2. AN EXAMPLE

Let  $M(Y, \sigma_Y)$  denote the collection of all  $\sigma_Y$ -invariant Borel probability measures on  $Y$ . Notice that

$$P(\phi, \psi) = \sup_{\nu \in \mathcal{M}(Y, \sigma_Y)} P(\phi, \psi, \nu),$$

where

$$P(\phi, \psi, \nu) = \frac{h_\nu(\sigma_Y)}{\int \psi d\nu} + P(\phi, \nu), \quad P(\phi, \nu) = \sup_{\substack{\mu \in \mathcal{M}(X, \sigma_X), \\ \mu \circ \pi^{-1} = \nu}} \frac{h_\mu(\sigma_X) - h_\nu(\sigma_Y)}{\int \phi d\mu}$$

for  $\nu \in M(Y, \sigma_Y)$ . Since  $\phi(x)$  and  $\psi(y)$  only depend on the first coordinates of  $x$  and  $y$ ,  $P(\phi, \psi, \nu)$  can only be maximized at Bernoulli measures  $\nu$  in  $M(Y, \sigma_Y)$ .

We make the following assumptions:

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<sup>1</sup>As a related result, Luzia [12] proved recently that the supremum in (1.1) always can be attained at ergodic measures when  $\phi$  and  $\psi$  are assumed to be general positive Hölder continuous functions.

- (1)  $\phi(x) \equiv \lambda > 0$  on  $X$  for some constant  $\lambda$ ;  
(2)  $\mathcal{B} = \{a, b\}$ ,  $\mathcal{A} = \{1, \dots, \ell_a, \ell_a + 1, \dots, \ell_a + \ell_b\}$ ,  $\tilde{\pi}(\{1, \dots, \ell_a\}) = \{a\}$ ,  $\tilde{\pi}(\{\ell_a + 1, \dots, \ell_a + \ell_b\}) = \{b\}$ , where  $\ell_a, \ell_b \in \mathbb{N}$ .

Then, since due to our assumption we have  $\int \phi d\mu = \lambda$  for all  $\mu \in M(X, \sigma_X)$ , the Ledrappier-Walters relativized variational principal [10] yields

$$P(\phi, \nu) = \frac{1}{\lambda}(\log(\ell_a)\nu([a]) + \log(\ell_b)\nu([b])),$$

where  $[c] := \{y = (y_i)_{i=1}^\infty \in Y : y_1 = c\}$  for  $c \in \mathcal{B}$ . Setting  $x = \nu([a])$  and  $H(x) = -x \log(x) - (1-x) \log(1-x)$ , we thus have for all Bernoulli measures  $\nu \in M(Y, \sigma_Y)$ ,

$$(2.1) \quad P(\phi, \psi, \nu) = f(x) = \frac{1}{\lambda}(\log(\ell_a/\ell_b)x + \log(\ell_b)) + \frac{H(x)}{\psi_a x + \psi_b(1-x)},$$

where  $\psi_a$  and  $\psi_b$  stand for the constant values of  $\psi$  over  $[a]$  and  $[b]$  respectively.

A counter-example will appear if we find  $\lambda$ ,  $\ell_a$ ,  $\ell_b$ ,  $\psi_a$ , and  $\psi_b$  such that  $f$  attains its maximum for at least two values of  $x$  in  $[0, 1]$ .

Setting  $U = \frac{\psi_b}{\lambda} \log(\ell_a/\ell_b)$  and  $V = \frac{\psi_a - \psi_b}{\psi_b}$ , the problem transfers to finding  $U \in \mathbb{R}$ ,  $V \in (-1, \infty)$  and  $M \geq 0$  such that

$$g(x) = Ux - M + \frac{H(x)}{1 + Vx} \leq 0, \quad \forall x \in [0, 1]$$

and  $g(x) = 0$  has more than one solution in  $[0, 1]$ . We can seek for a quadratic polynomial  $F(x) = A - B(x - 1/2)^2$  with  $A, B > 0$  such that

- (i)  $F(x) \geq H(x)$  for all  $x \in [0, 1]$ ; and
- (ii) the equation  $F(x) = H(x)$  has more than one solution in  $[0, 1]$ .

Due to the common symmetry properties of  $F$  and  $H$  with respect to  $x = 1/2$  and the concavity of these functions, this will be the case if we make sure that the curvature of  $F$  at  $1/2$  is larger than that of  $H$  at  $1/2$  and  $\inf_{x \in [0, 1]} (F(x) - H(x)) = 0$ . Recalling that the curvature of a smooth function  $h(x)$  being given by

$$\mathcal{K}_h(x) = \frac{|h''(x)|}{(1 + (h'(x))^2)^{3/2}},$$

we have  $\mathcal{K}_H(1/2) = 4$  and  $\mathcal{K}_F(1/2) = 2B$ . Thus we get the following necessary and sufficient condition to guarantee that (i)-(ii) hold:

$$(2.2) \quad B > 2, \quad A = \max_{0 \leq x \leq 1} (B(x - 1/2)^2 + H(x)).$$

Now take a pair of numbers  $A, B$  so that (2.2) holds. Then the identity

$$-(Ux - M)(1 + Vx) = A - B(x - 1/2)^2$$

yields

$$\begin{cases} UV = B, \\ MV - U = B, \\ M = A - B/4. \end{cases}$$

This forces

$$(A - B/4)V^2 - BV - B = 0.$$

The positive root of the above equation is

$$(2.3) \quad V = \frac{2B + 4\sqrt{AB}}{4A - B}.$$

Then, using the equality  $UV = B$  yields

$$(2.4) \quad U = \sqrt{AB} - B/2.$$

Next take

$$(2.5) \quad \psi_b = 1, \quad \psi_a - \psi_b = V,$$

and take positive integers  $\ell_a, \ell_b$  such that

$$(2.6) \quad \log(\ell_a/\ell_b) > \frac{1+V}{V}B.$$

In the end, take  $\lambda$  such that

$$(2.7) \quad \frac{\log(\ell_a/\ell_b)}{\lambda} = U, \text{ i.e., } \lambda = \frac{\log(\ell_a/\ell_b)}{U} = \log(\ell_a/\ell_b) \frac{V}{B}.$$

According to (2.6)-(2.7),  $\lambda > 1 + V$  and thus  $\phi \equiv \lambda > \max(\psi) = \max(\psi_a, \psi_b)$ .

Then for the above constructed  $\lambda, \ell_a, \ell_b, \psi_a$ , and  $\psi_b$ , the function  $f(x)$  defined in (2.1) attains its supremum at two different points  $x$  in  $[0, 1]$ . This yields an example that the supremum in (1.1) is attained at two different Bernoulli measures in  $M(X, \sigma_X)$ .

In the end, we provide a more concrete example for  $\lambda, \ell_a, \ell_b, \psi_a$ , and  $\psi_b$ .

**Example 2.1.** *Set*

$$B = 3 \log 2 \approx 2.07944$$

*and*

$$A = \log 3 - \frac{7}{12} \log 2 \approx 0.69427643.$$

*One can check that (1.1) holds for such  $A$  and  $B$ . Indeed, the supremum in defining  $A$  is attained at  $x = 1/3$ . Then*

$$U = \sqrt{AB} - B/2 \approx 0.16182292, \quad V = \frac{2B + 4\sqrt{AB}}{4A - B} \approx 12.8501046.$$

*Take*

$$\psi_a = 1 + V \approx 13.8501046, \quad \psi_b = 1$$

*and*

$$\ell_a = 150, \quad \ell_b = 1, \quad \lambda = \log(\ell_a/\ell_b) \cdot \frac{V}{B} \approx 30.9636922.$$

### 3. APPLICATION TO GATZOURAS-LALLEY CARPETS

Let  $\lambda$ ,  $\ell_a$ ,  $\ell_b$ ,  $\psi_a$ , and  $\psi_b$  be constructed as in Example 2.1. Notice that

$$3 \exp(-\lambda) \ell_a < 3 \cdot e^{-30} \cdot 150 < 1, \quad \exp(-\psi_a) + \exp(-\psi_b) < 2e^{-1} < 1.$$

Then we can build a Gatzouras-Lalley carpet in the unit square as the attractor  $K$  of the IFS  $\{S_{a,r} : 1 \leq r \leq \ell_a\} \cup \{S_{b,s} : 1 \leq s \leq \ell_b\}$ , where

$$\begin{cases} S_{a,r}(x, y) = (\exp(-\lambda)x, \exp(-\psi_a)y) + (2r \exp(-\lambda), 0), & 1 \leq r \leq \ell_a, \\ S_{b,s}(x, y) = (\exp(-\lambda)x, \exp(-\psi_b)y) + (2r \exp(-\lambda), 1 - \exp(-\psi_b)), & 1 \leq s \leq \ell_b. \end{cases}$$

Gatzouras and Lalley [6] proved that the Hausdorff dimension of  $K$  is equal to  $P(\phi, \psi)$ , which is attained by some Bernoulli measure on  $X$ . The previous section shows that such a measure is not unique; in our example there are exactly two such measures.

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LAGA (UMR 7539), DÉPARTEMENT DE MATHÉMATIQUES, INSTITUT GALILÉE, UNIVERSITÉ PARIS 13,  
99 AVENUE JEAN-BAPTISTE CLÉMENT , 93430 VILLETANEUSE, FRANCE

*E-mail address:* `barral@math.univ-paris13.fr`

DEPARTMENT OF MATHEMATICS, THE CHINESE UNIVERSITY OF HONG KONG, SHATIN, HONG KONG,

*E-mail address:* `djfeng@math.cuhk.edu.hk`